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THE USE OF CYCLIC SYMMETRY IN TWO-DIMENSIONAL ELASTIC STRESS ANALYSIS BY BEM

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Abstract—Rotationally periodic (or cyclic) symmetry is exploited in the elastic stress analysis of two-dimensional structures under arbitrary load conditions by the BEM. It is proved that the coefficient matrices of the global boundary element equations for the rotationally periodic system are block-circulant so long as a kind of symmetry-adapted reference coordinate system is adopted. Furthermore, the computational convenience and efficiency which can be achieved by exploiting this property and the structural geometric symmetry in the different phases of numerical implementation is demonstrated, and an efficient algorithm is presented. Numerical examples are given to illustrate the advantages of such exploitation of symmetry in the context of the BEM.

INTRODUCTION

The boundary element method (BEM) is a numerical method based on integral formulations, which offers several important advantages over 'domain' type techniques such as the finite element method (FEM) and the finite difference method (FDM). However, compared with the FEM, the BEM deals with asymmetric coefficient matrices, and the calculations of the coefficients are relatively complicated.

The exploitation of symmetry in structural static, dynamic and stability analyses by using the FEM has been presented in many references, see Thomas (1979), Healey (1988), Wu (1988), Dinkevich (1991), Liu and Wu (1993) amongst others, and now it is common practice to consider the use of symmetry in engineering analysis problems. In the context of the BEM, the exploitation of axisymmetry and some simple reflection symmetry cases has been demonstrated by Rizzo and Shippy (1979), Mayr *et al.* (1980), Crouch and Starfield (1983), Manolis and Beskos (1988), Saigal *et al.* (1990) etc., where many advantages in convenience, accuracy and efficiency have been observed. However for another class of symmetric engineering structures such as cyclic (rotationally periodic) structures and regular-polygonal structures which have been well investigated by using the FEM, except the work of Maier *et al.* (1983), there is an evident deficiency of the BEM approach, in particular from the application point of view.

The present paper is a further effort to make up such deficiency, and it is expected that it will bring the exploitation of symmetry into more practical use in the engineering design practice by using the BEM. For these purposes, instead of considering more general cases, this paper simply uses the two-dimensional elastic stress analysis problem and two practical examples to demonstrate the full advantages of the use of symmetry in the BEM. The contributions of the present paper are twofold. Firstly, instead of assuming the special properties of the system matrices of the BEM equations for cyclic structures, it adopts a simple, direct and explicit way to prove the block-circulant property of the coefficient matrices which the systems enjoy. This approach, together with those basic concepts introduced, will enable the further exploitation of other types of symmetry. Secondly, the computational convenience and efficiency of such exploitation are fully discussed and demonstrated by means of two numerical examples.

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ROTATIONALLY PERIODIC SYMMETRY AND COMPUTATIONAL MODEL

A structure or a computational region Ω is said to possess rotationally periodic symmetry of order N when its geometry, physical properties and constraint conditions are invariant under the following N symmetry transformations (or operations)

$$\phi_k = (2\pi/N)(k-1) \quad (k = 1, \ldots, N), \tag{1}$$

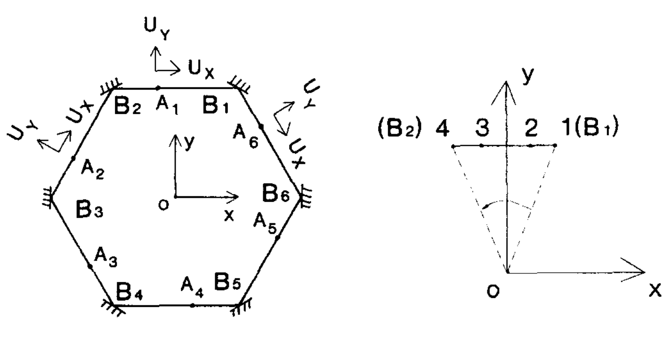
where ϕ_k represents a rotation of Ω about its axis of rotation with an angle of $2(k-1)\pi/N$. The difference between axisymmetry and rotationally periodic symmetry is that, with axisymmetry a structure (or a region) can rotate any angle about its axis of rotation without change, but with rotationally periodic symmetry a structure (or region) can only rotate N different angles without change, see eqn (1). It is convenient that the axis of rotation is defined as the Z-axis in a rectangular or cylindrical coordinate system. For example, Fig. 1(a) shows a plane region possessing rotationally periodic symmetry of order $N = 6$; Fig. 2 shows a plane infinite region with an internal square hole, which can be regarded as a rotationally periodic system of symmetry order $N = 4$.

When the BEM is employed to analyze a rotationally periodic system Ω, to make full use of its symmetry the following two fundamental points, which are also the basic requirements of the method given in this paper, should be followed. The first one is to discretize the computational boundary in a symmetric way such that the boundary element (BE) discretization model of Ω keeps the original symmetry of the system. The second one is to adopt a symmetry-adapted coordinate system, which will be called "symmetric coordinate system", as a reference system for nodal displacements and forces on the boundary. To these ends, the next paragraph briefly shows how to form a BE computational model of symmetric system.

Designating the boundary of Ω as C, it is obvious that C can be naturally divided into N identical parts, which will be called symmetric regions in this paper. Ordering these N parts in anticlockwise sequence, and designating them as C_k $(k = 1, \ldots, N)$, it follows that:

$$C_k = \phi_k : C_1; \quad C = C_1 \cup C_2 \cup \cdots \cup C_N. \tag{2}$$

This equation means that C_k can be obtained from C_1, which is called "basic region" and can be arbitrarily selected from those identical parts, by the application of the symmetry operation ϕ_k, and all these N different C_k $(k = 1, \ldots, N)$ cover C. Discretizing C_1 only, one can then obtain the discretization model of C by using eqn (2), which satisfies the above first requirement. For any node A_1 in the basic region, there are certainly another $N-1$ different nodes which are located symmetrically on the other $N-1$ symmetric regions. All



(a) (b)

Fig. 1. A rotationally periodic plane plate with $N = 6$. (a) The symmetric node orbit O_A and its corresponding symmetric coordinate system for reference; (b) the basic region.

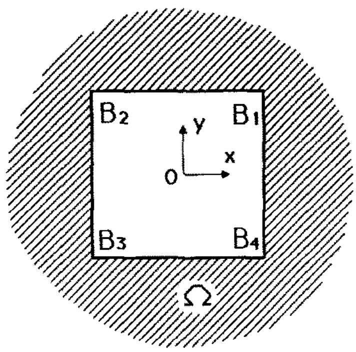


Fig. 2. An infinite plate with a square hole at the center.

these N nodes constitute a set of symmetric nodes, which is called "symmetric node orbit" (or just "orbit") and is designated as O_A

$$O_A = \{A_1, A_2, \ldots, A_N\}. \tag{3}$$

For the N nodes of O_A, the reference coordinate directions of node A_1 which belongs to the basic region are first defined, then the reference coordinate directions for the other $N-1$ nodes can be obtained from those of node A_1 through the last $N-1$ different symmetry transformations of eqn (1), see Fig. 1(a) for example. If the line B_1–B_2 shown in Fig. 1(b) is regarded as the basic region C_1 for the computational boundary of Fig. 1(a), because of cyclic symmetry of the computational model, it is readily seen that the two interface nodes of C_1 belong to the same orbit O_B, i.e. $B_1 \subset O_B$ and $B_2 \subset O_B$. In the following, only those nodes which are located on the internal part of C_1 and the "right" interface of C_1 are regarded as belonging to the basic region. For example, as the two interface nodes of the basic region C_1 in Fig. 1(b), only B_1 will be regarded as belonging to C_1 and B_2 will then be regarded as belonging to C_2; in the case of four computational nodes being used on the line B_1–B_2, only three nodes $\{B_1 = 1, 2, 3\}$ will be regarded as belonging to C_1. If the number of the nodes belonging to C_1 is denoted as M, then the total computational nodes on C will be NM.

It should be mentioned that, the introduction of some basic concepts in this section (e.g. symmetric coordinate system, symmetric node orbit etc.) is not only for the convenience of the following discussions but will also benefit the exploitation of general symmetry cases as it has in the finite element (FE) analysis of general symmetric structures, see Wu (1988).

THE PROPERTIES OF THE GLOBAL COEFFICIENT MATRICES

Based on Somigliana's identity, the BE equations can be expressed as:

$$\mathbf{HU} = \mathbf{GF} + \mathbf{B}, \tag{4}$$

where $\mathbf{U}$ and $\mathbf{F}$ are the vectors of the displacements and tractions on all the nodes of C, respectively, $\mathbf{H}$ and $\mathbf{G}$ are the global coefficient matrices of the BE system and $\mathbf{B}$ is the vector accounting for the body forces.

Consider a rotationally periodic system. By means of the concept of symmetric regions, $\mathbf{U}$ and $\mathbf{F}$ can be written as

$$\mathbf{U}^T = [\mathbf{U}^{1T}, \mathbf{U}^{2T}, \ldots, \mathbf{U}^{NT}], \tag{5}$$

$$\mathbf{F}^T = [\mathbf{F}^{1T}, \mathbf{F}^{2T}, \ldots, \mathbf{F}^{NT}], \tag{6}$$

in which $\mathbf{U}^k$ is a vector which collects the displacements on the nodes belonging to kth symmetric region, and $\mathbf{F}^k$ represents the corresponding traction vector. In such an ordering way, eqn (4) can be written as:

$$\begin{bmatrix} \mathbf{H}^{11} & \mathbf{H}^{12} & \dots & \mathbf{H}^{1N} \\ \mathbf{H}^{21} & \mathbf{H}^{22} & \dots & \mathbf{H}^{2N} \\ \vdots & \vdots & & \vdots \\ \mathbf{H}^{N1} & \mathbf{H}^{N2} & \dots & \mathbf{H}^{NN} \end{bmatrix} \begin{Bmatrix} \mathbf{U}^1 \\ \mathbf{U}^2 \\ \vdots \\ \mathbf{U}^N \end{Bmatrix} = \begin{bmatrix} \mathbf{G}^{11} & \mathbf{G}^{12} & \dots & \mathbf{G}^{1N} \\ \mathbf{G}^{21} & \mathbf{G}^{22} & \dots & \mathbf{G}^{2N} \\ \vdots & \vdots & & \vdots \\ \mathbf{G}^{N1} & \mathbf{G}^{N2} & \dots & \mathbf{G}^{NN} \end{bmatrix} \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \\ \vdots \\ \mathbf{F}^N \end{Bmatrix} + \begin{Bmatrix} \mathbf{B}^1 \\ \mathbf{B}^2 \\ \vdots \\ \mathbf{B}^N \end{Bmatrix}. \tag{7}$$

The properties of the global coefficient matrices **H** and **G** in eqn (7) will be examined in the following, note that the symmetric coordinate system is adopted as the reference system for the components of **U** and **F**. To this end, the influence coefficients of a rotationally periodic system will be studied first below.

The displacement component u_i^Q at a field point Q in the i-direction, which is excited by a concentrated force f_j^P applied at a load point P in the j-direction, is:

$$u_i^Q = G_{ij}^{QP} f_j^P, \tag{8}$$

where G_{ij}^{QP} is the displacement influence coefficient (Kelvin's singular solution) which can be expressed, for example, for the two-dimensional plane strain problem in a rectangular Cartesian coordinate system X–Y as:

$$G_{ij}^{QP} = \frac{1}{2G}\left[(3-4\upsilon)g\delta_{ij} - (x_j - c_j)\frac{\partial g}{\partial x_i}\right], \tag{9}$$

in which

$$g(x, y) = \frac{-1}{4\pi(1-\upsilon)} \ln\left[(x-c_x)^2 + (y-c_y)^2\right]^{1/2}. \tag{10}$$

In eqns (8)–(10), G is the shear modulus; υ is the Poisson's ratio; δ_{ij} is the Kronecker delta; c_x and c_y are the X- and Y-projections of the distance between points P and Q, respectively; $i, j = x, y$; and x_x is x, x_y is y. Observing Fig. 3, let $\bar{P}$ and $\bar{Q}$ be another two points on the boundary, which come from P and Q through a symmetry transformation

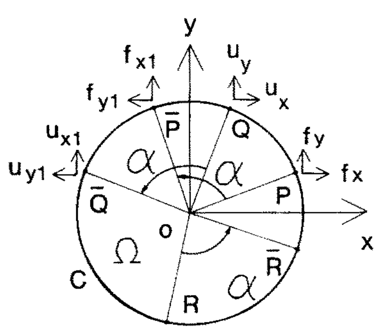


Fig. 3. Illustration of the load points and the field points on the boundary contour C of a symmetric domain Ω and their reference directions: P, $\bar{P}$ load points; Q, $\bar{Q}$ field points; R; $\bar{R}$ integration points.

(rotation about the axis of rotation) $\alpha = \phi_k$ ($k = 1, \ldots, N$), respectively. Similarly, corresponding to the integration point R which has coordinates x and y, another integration point $\bar{R}$ can be obtained from R through the same symmetry transformation $\alpha = \phi_k$. Designating the coordinates of $\bar{R}$ as $\bar{x}$ and $\bar{y}$, it follows:

$$\bar{x} = x \cos \alpha - y \sin \alpha, \quad \bar{y} = x \sin \alpha + y \cos \alpha, \tag{11}$$

$$\bar{c}_x = c_x \cos \alpha - c_y \sin \alpha, \quad \bar{c}_y = c_x \sin \alpha + c_y \cos \alpha, \tag{12}$$

where $\bar{c}_x$ and $\bar{c}_y$ are the projections of the distance between $\bar{P}$ and $\bar{Q}$. Note that in eqns (9) and (10) it has been assumed that the reference directions for points P and Q are the same as the X–Y coordinate system. For point $\bar{P}$, its reference directions are obtained from those of point P through the corresponding symmetry transformation; and for point $\bar{Q}$, its reference directions are obtained from Q by using the same rule. This in fact is the way of the symmetric coordinate system being obtained. If there are two concentrated force components $f_{x1}^{\bar{P}}$ and $f_{y1}^{\bar{P}}$ acting on point $\bar{P}$ and referring to its own reference directions, one can easily transform them into the components referring to the reference directions of point P:

$$f_x^{\bar{P}} = f_{x1}^{\bar{P}} \cos \alpha - f_{y1}^{\bar{P}} \sin \alpha; \quad f_y^{\bar{P}} = f_{x1}^{\bar{P}} \sin \alpha + f_{y1}^{\bar{P}} \cos \alpha. \tag{13}$$

Using eqns (8)–(10), the displacement components $u_x^{\bar{Q}}$ and $u_y^{\bar{Q}}$ of point $\bar{Q}$, which refer to the reference directions of point Q, can be obtained as:

$$\begin{aligned} u_x^{\bar{Q}} &= G_{xx}^{\bar{Q}\bar{P}}(\bar{x}, \bar{y}) f_x^{\bar{P}} + G_{xy}^{\bar{Q}\bar{P}}(\bar{x}, \bar{y}) f_y^{\bar{P}}, \\ u_y^{\bar{Q}} &= G_{yx}^{\bar{Q}\bar{P}}(\bar{x}, \bar{y}) f_x^{\bar{P}} + G_{yy}^{\bar{Q}\bar{P}}(\bar{x}, \bar{y}) f_y^{\bar{P}}, \end{aligned} \tag{14}$$

in which the displacement influence coefficients are referring to the integration point $\bar{R}$ and given by:

$$G_{ij}^{\bar{Q}\bar{P}}(\bar{x}, \bar{y}) = \frac{1}{2G}\left[(3-4\upsilon)\bar{g}\delta_{ij} - (x_j - \bar{c}_j)\frac{\partial \bar{g}}{\partial x_i}\right] x = \bar{x}, y = \bar{y}, \tag{15}$$

and

$$\bar{g}(x, y) = \frac{-1}{4\pi(1-\upsilon)} \ln [(x-\bar{c}_x)^2 + (y-\bar{c}_y)^2]^{1/2}. \tag{16}$$

Transforming the displacement components $u_x^{\bar{Q}}$ and $u_y^{\bar{Q}}$ into components $u_{x1}^{\bar{Q}}$ and $u_{y1}^{\bar{Q}}$, which refer to the reference directions of point $\bar{Q}$, and noting that the reference directions of point $\bar{Q}$ are obtained from those of point Q by a symmetric rotation $\alpha = \phi_k$, one has:

$$u_{x1}^{\bar{Q}} = u_x^{\bar{Q}} \cos \alpha + u_y^{\bar{Q}} \sin \alpha; \quad u_{y1}^{\bar{Q}} = -u_x^{\bar{Q}} \sin \alpha + u_y^{\bar{Q}} \cos \alpha. \tag{17}$$

Substituting eqns (11)–(16) into (17), and only taking one of the two force components into account, it follows:

$$u_{i1}^{\bar{Q}} = G_{i1,j1}^{\bar{Q}\bar{P}}(\bar{x}, \bar{y}) f_{j1}^{\bar{P}} = G_{ij}^{QP}(x, y) f_{j1}^{\bar{P}} \tag{18}$$

in which $G_{i1,j1}^{\bar{Q}\bar{P}}(\bar{x}, \bar{y})$ is the displacement influence coefficient relating points $\bar{P}$ and $\bar{Q}$, and it refers to the integration point $\bar{R}$. This influence coefficient specifies the displacement at field point $\bar{Q}$ in the $i1$-direction of the reference directions of point $\bar{Q}$, due to a unit concentrated force applied at load point $\bar{P}$ in the $j1$-direction of the reference directions of point $\bar{P}$. Equation (18) shows that $G_{i1,j1}^{\bar{Q}\bar{P}}(\bar{x}, \bar{y})$ is equal to $G_{ij}^{QP}(x, y)$. It is readily seen that although the previous discussion assumed that points P and Q have the same reference directions, it is not difficult to prove that eqn (18) still applies if P and Q have different reference directions so long as the reference directions for points $\bar{P}$ and $\bar{Q}$ are adopted in the way described above. From eqn (18) one can conclude that, for a rotationally periodic system, the influence coefficients corresponding to any two symmetric sets of points and referring to two symmetric integration points, respectively, are equal if a symmetry-adapted system is used, i.e. the influence coefficients have the same symmetric properties as the system itself under a symmetry-adapted system.

Since the coefficients in eqns (8) and (18) comprise the fundamental solution for the two-dimensional elastostatic problem, and because of the properties of the fundamental solution under the symmetric coordinate system and the symmetric form of the boundary element discretization model, it can be readily verified from the boundary element procedure that, if the displacement and traction vectors for kth symmetric region ($k = 1, \ldots, N$) are ordered in the following way:

$$\mathbf{U}^k = [u_{x1}^k, u_{y1}^k, u_{x2}^k, u_{y2}^k, \ldots, u_{xM}^k, u_{yM}^k]^T;$$
$$\mathbf{F}^k = [f_{x1}^k, f_{y1}^k, f_{x2}^k, f_{y2}^k, \ldots, f_{xM}^k, f_{yM}^k]^T \tag{19}$$

where all the components are referring to the symmetric coordinate system, then matrix G in eqn (7) satisfies the relation:

$$\mathbf{G}^{ij} = \mathbf{G}^{i+k,j+k} \quad (i, j = 1, \ldots, N;\ k = 1, \ldots, N)$$
$$(\text{if } i+k > N, \text{ it reads as } i+k-N\text{; similarly for } j+k) \tag{20}$$

i.e.

$$\mathbf{G}^{11} = \mathbf{G}^{22}, \mathbf{G}^{12} = \mathbf{G}^{23}, \ldots, \mathbf{G}^{1,N-1} = \mathbf{G}^{2N}, \mathbf{G}^{1N} = \mathbf{G}^{21};$$
$$\mathbf{G}^{11} = \mathbf{G}^{33}, \mathbf{G}^{12} = \mathbf{G}^{34}, \ldots, \mathbf{G}^{1,N-1} = \mathbf{G}^{31}, \mathbf{G}^{1N} = \mathbf{G}^{32}; \ldots \tag{21}$$

Therefore it is proved that matrix **G** is a block-circulant matrix. In the same way one can reach the conclusion that matrix **H** is also a block-circulant matrix. Designating

$$\mathbf{G}^{1k} = \mathbf{G}^k, \quad \mathbf{H}^{1k} = \mathbf{H}^k (k = 1, \ldots, N) \tag{22}$$

matrices **G** and **H** in eqn (7) can be written as:

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}^1\mathbf{G}^2 \ldots \ \mathbf{G}^N \\ \mathbf{G}^N\mathbf{G}^1 \ldots \mathbf{G}^{N-1} \\ \vdots \quad \vdots \qquad \vdots \\ \mathbf{G}^2\mathbf{G}^3 \ldots \ \mathbf{G}^1 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} \mathbf{H}^1\mathbf{H}^2 \ldots \ \mathbf{H}^N \\ \mathbf{H}^N\mathbf{H}^1 \ldots \mathbf{H}^{N-1} \\ \vdots \quad \vdots \qquad \vdots \\ \mathbf{H}^2\mathbf{H}^3 \ldots \ \mathbf{H}^1 \end{bmatrix}. \tag{23}$$

Note that, unlike the stiffness matrix of FE equations, in general these two matrices are not symmetric matrices.

PARTITIONING ALGORITHM

From the concept of the symmetric node orbit O_A, it is readily seen that the displacement vector $\mathbf{d}_A$

$$\mathbf{d}_A = [d_A^1, d_A^2, \ldots, d_A^N]^T \ (d \text{ may be one of } u_x \text{ and } u_y) \tag{24}$$

where d_A^k ($k = 1, \ldots, N$) represents the displacement of node A_k which belongs to the orbit O_A and is located on the kth symmetric region, constructs an invariant subspace in the whole displacement space under all symmetry transformations. Using the following N complete symmetrized basis vectors:

$$\begin{aligned}
\mathbf{e}_1 &= \mathbf{C}_O = \sqrt{1/N}[1,1,\ldots,1]^T;\\
\mathbf{e}_{2m} &= \mathbf{C}_m = \sqrt{2/N}[\cos m\theta_1, \cos m\theta_2, \ldots, \cos m\theta_N]^T,\\
\mathbf{e}_{2m+1} &= \mathbf{S}_m = \sqrt{2/N}[\sin m\theta_1, \sin m\theta_2, \ldots, \sin m\theta_N]^T,\\
&(m = 1,\ldots,[(N-1)/2];\quad \theta_k = 2\pi(k-1)/N,\quad k = 1,\ldots,N);\\
\mathbf{e}_N &= \mathbf{C}_{N/2} = \sqrt{1/N}[1,-1,1,-1,\ldots,1,-1]^T \quad \text{(when } N \text{ is even)}
\end{aligned} \tag{25}$$

where $[(N-1)/2]$ is the largest integer which does not exceed $(N-1)/2$, the vector $\mathbf{d}_A$ can be expanded as:

$$\mathbf{d}_A = \sum_{i=1}^{N} \tilde{d}_A^i \mathbf{e}_i = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \ldots \quad \mathbf{e}_N]\tilde{\mathbf{d}}_A \tag{26}$$

in which $\tilde{d}_A^i$ $(i = 1,\ldots,N)$ is the expansion coefficient of the displacement vector $\mathbf{d}_A$ on the ith new basis vector $\mathbf{e}_i$, and it is termed generalized displacement in this paper. As all the new basis vectors construct a set of orthogonal unit vectors, therefore it can be obtained from eqn (26) that

$$\tilde{\mathbf{d}}_A = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \ldots \quad \mathbf{e}_N]^T \mathbf{d}_A. \tag{27}$$

Similarly, for the traction vector $\mathbf{t}_A$ of the orbit O_A

$$\mathbf{t}_A = [t_A^1, t_A^2, \ldots, t_A^N]^T \quad (t \text{ may be one of } f_x \text{ and } f_y) \tag{28}$$

one has:

$$\mathbf{t}_A = \sum_{i=1}^{N} \tilde{t}_A^i \mathbf{e}_i = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \ldots \quad \mathbf{e}_N]\tilde{\mathbf{t}}_A \tag{29}$$

$$\tilde{\mathbf{t}}_A = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \ldots \quad \mathbf{e}_N]^T \mathbf{t}_A \tag{30}$$

where $\tilde{t}_A^i$ will be called generalized traction. Thus using eqns (26) and (29) for all the orbits, one can obtain:

$$\mathbf{U} = \mathbf{R}\tilde{\mathbf{U}}; \quad \mathbf{F} = \mathbf{R}\tilde{\mathbf{F}}; \quad \mathbf{R} = [e_{ij} \quad \mathbf{I}]^T \quad (i,j = 1,\ldots,N), \tag{31}$$

where $\mathbf{I}$ is a $2M$-dimensional unit matrix, e_{ij} is the jth element of the basis vector $\mathbf{e}_i$, $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{F}}$ are called global generalized displacement and traction vectors, respectively,

$$\begin{aligned}
\tilde{\mathbf{U}} &= [\tilde{\mathbf{U}}^1, \tilde{\mathbf{U}}^2, \ldots, \tilde{\mathbf{U}}^N];\\
\tilde{\mathbf{U}}^i &= [\tilde{u}_{x1}^i, \tilde{u}_{y1}^i, \tilde{u}_{x2}^i, \tilde{u}_{y2}^i, \ldots, \tilde{u}_{xM}^i, \tilde{u}_{yM}^i]
\end{aligned} \tag{32}$$

and $\tilde{\mathbf{F}}$ has a similar form. Left-multiplying the two sides of eqn (7) by $\mathbf{R}^T$, and substituting (31) into (7), one has:

$$\tilde{\mathbf{H}}\tilde{\mathbf{U}} = \tilde{\mathbf{G}}\tilde{\mathbf{F}} + \tilde{\mathbf{B}} \tag{33}$$

in which

$$\tilde{\mathbf{H}} = \mathbf{R}^T\mathbf{H}\mathbf{R}; \quad \tilde{\mathbf{G}} = \mathbf{R}^T\mathbf{G}\mathbf{R}; \quad \tilde{\mathbf{B}} = \mathbf{R}^T\mathbf{B}. \tag{34}$$

Substituting eqns (23) and (31) into (34), one has:

$$\tilde{\mathbf{H}} = [\tilde{\mathbf{H}}^{ij}]; \quad \tilde{\mathbf{G}} = [\tilde{\mathbf{G}}^{ij}]; \quad \tilde{\mathbf{B}} = [\tilde{\mathbf{B}}^{1T}, \tilde{\mathbf{B}}^{2T}, \ldots, \tilde{\mathbf{B}}^{NT}]^T;$$

$$\tilde{\mathbf{H}}^{ij} = \sum_{l=1}^{N} \mathbf{H}^l \left[\sum_{k=1}^{N} (e_{1,k+N-l+1})(e_{jk}) \right] \quad (i, j = 1, \ldots, N);$$

$$\tilde{\mathbf{B}}^i = \sum_{k=1}^{N} e_{ik} \mathbf{B}^k \quad (i = 1, \ldots, N) \tag{35}$$

and $\tilde{\mathbf{G}}^{ij}$ has the same form as $\tilde{\mathbf{H}}^{ij}$. Note that if the second subscript $k+N-l+1$ in $e_{i,k+N-l+1}$ is greater than N, it will be read as $k-l+1$. Further by using eqn (25) it can be obtained that:

$$\tilde{\mathbf{H}} = \sum \oplus \tilde{\mathbf{H}}_{mm}; \quad \tilde{\mathbf{G}} = \sum \oplus \tilde{\mathbf{G}}_{mm} \quad (m = 0, \ldots, [N/2]) \tag{36}$$

in which $\oplus$ represents the direct sum of matrices, i.e. the matrices $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{G}}$ have block-diagonal form, and

$$\tilde{\mathbf{H}}_{OO} = \tilde{\mathbf{H}}^{11} = (\mathbf{H}^1 + \mathbf{H}^2 + \cdots + \mathbf{H}^N) \tag{37}$$

$$\tilde{\mathbf{H}}_{mm} = \begin{bmatrix} \tilde{\mathbf{H}}^{cc}_{mm} & \tilde{\mathbf{H}}^{cs}_{mm} \\ \tilde{\mathbf{H}}^{sc}_{mm} & \tilde{\mathbf{H}}^{ss}_{mm} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{H}}^{pp} & \tilde{\mathbf{H}}^{pq} \\ \tilde{\mathbf{H}}^{qp} & \tilde{\mathbf{H}}^{qq} \end{bmatrix};$$

$$\tilde{\mathbf{H}}^{pp} = \tilde{\mathbf{H}}^{qq} = \sum_{l=1}^{N} \mathbf{H}^l \cos(l-1)m\beta; \quad \tilde{\mathbf{H}}^{pq} = -\tilde{\mathbf{H}}^{qp} = \sum_{l=1}^{N} \mathbf{H}^l \sin(l-1)m\beta$$

$$(\beta = 2\pi/N; \quad p = 2m, q = 2m+1; \, m = 1, \ldots, [(N-1)/2]) \tag{38}$$

$$\tilde{\mathbf{H}}_{N/2,N/2} = \tilde{\mathbf{H}}^{NN} = (\mathbf{H}^1 - \mathbf{H}^2 + \mathbf{H}^3 - \mathbf{H}^4 + \cdots - \mathbf{H}^N) \quad (\text{when } N \text{ is even}), \tag{39}$$

$\tilde{\mathbf{G}}_{mm}$ ($m = 0, \ldots, [N/2]$) have the same form as $\tilde{\mathbf{H}}_{mm}$. Based on eqn (36), it is obvious that the solution problem of eqn (33) can be naturally partitioned into $[(N+2)/2]$ decoupled subproblems:

$$\tilde{\mathbf{H}}_{mm}\tilde{\mathbf{U}}_m = \tilde{\mathbf{G}}_{mm}\tilde{\mathbf{F}}_m + \tilde{\mathbf{B}}_m \quad (m = 0, \ldots, [N/2]) \tag{40}$$

where

$$\tilde{\mathbf{U}}_O = \tilde{\mathbf{U}}^1; \quad \tilde{\mathbf{U}}_{N/2} = \tilde{\mathbf{U}}^N \quad (\text{when } N \text{ is even});$$

$$\tilde{\mathbf{U}}_m = [\tilde{\mathbf{U}}^{pT}, \tilde{\mathbf{U}}^{qT}]^T \quad (p = 2m, q = 2m+1; \, m = 1, \ldots, [(N-1)/2]) \tag{41}$$

and $\tilde{\mathbf{F}}_m$, $\tilde{\mathbf{B}}_m$ have the same form as $\tilde{\mathbf{U}}_m$.

Therefore instead of solving the original system eqn (7), now one only needs to solve a series of independent small subproblems as shown in eqn (40). Obviously the partitioning of the original problem into a series of small subproblems will lead to a high efficiency of computation which will also be demonstrated by the numerical examples given in the next section.

Consider a special case in which the given load distributions and given displacement conditions have the same rotationally periodic symmetry as the system. In this case, it is obvious that all the body force subvectors in eqn (7) satisfy:

$$\mathbf{B}^1 = \mathbf{B}^2 = \cdots = \mathbf{B}^N \tag{42}$$

so that from eqn (35) it follows:

$$\tilde{\mathbf{B}}^2 = \tilde{\mathbf{B}}^3 = \cdots = \tilde{\mathbf{B}}^N = \mathbf{0}, \quad \tilde{\mathbf{B}}^1 = \sqrt{N}\mathbf{B}^1. \tag{43}$$

Similarly, if $\mathbf{d}_A$ is given, one has

$$\tilde{d}_A^1 \neq 0, \quad \tilde{d}_A^i = 0 \quad (i = 2, \ldots, N) \tag{44}$$

and if $\mathbf{t}_A$ is given, it follows

$$\tilde{t}_A^1 \neq 0, \quad \tilde{t}_A^i = 0 \quad (i = 2, \ldots, N). \tag{45}$$

All these imply that except for the first subproblem, the other [N/2] subproblems have zero solutions, therefore one only needs to solve the first subproblem. Consider another special case for which the load distributions etc. have the following property:

$$\mathbf{B}^k = \mathbf{B}^1 \cos\theta_k \quad (k = 1, \ldots, N) \tag{46}$$

which, by using eqns (25) and (35), results in

$$\tilde{\mathbf{B}}^2 = (\sqrt{N/2})\mathbf{B}^1, \quad \tilde{\mathbf{B}}^i = \mathbf{0} \quad (i \neq 2) \tag{47}$$

and hence only the second subproblem needs to be solved. From the above discussions it can be deduced that the use of symmetry of the system will provide a special advantage for making use of the symmetry of the load conditions, and one need not introduce any special boundary conditions for such purpose, it makes the method presented here even more effective and convenient.

NUMERICAL EXAMPLES

The computational convenience and efficiency achieved in the analysis of rotationally periodic systems by using their symmetry are further demonstrated here through the numerical solution of the following two nontrivial two-dimensional elastostatic problems, which also verify the generality of the method. For these problems, the Poisson's ratio is 0.2 and the Young's modulus is 0.7×10^5 psi. All the examples were solved by using the direct BE algorithm and constant elements on a HP-9000/870 computer.

1. *A disk with eight symmetrical holes under a torsion*

Figure 4 shows the dimensions of the structure, the load distribution and the constraint conditions. Figure 5(a) shows the basic symmetric part, in which dashed lines are the inter-boundaries of the basic region with the adjacent symmetric parts, but only the solid curves belong to the computational boundaries and were meshed. The computed results were compared with a detailed FE analysis, and a very good agreement has been observed. Figure 5(b) shows the computed deformation of the inner boundary of the basic symmetric part. In order to show the effect of the symmetry order N to the computation efficiency, this problem was deliberately solved for the following three cases: case A, it was solved without using any symmetry properties; case B, its rotationally periodic symmetry of order

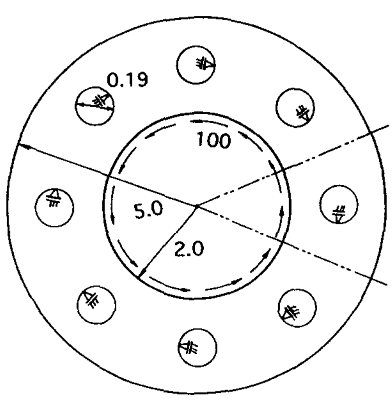
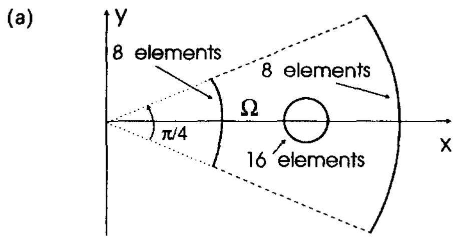


Fig. 4. A disk with eight symmetrical holes under a torsion.



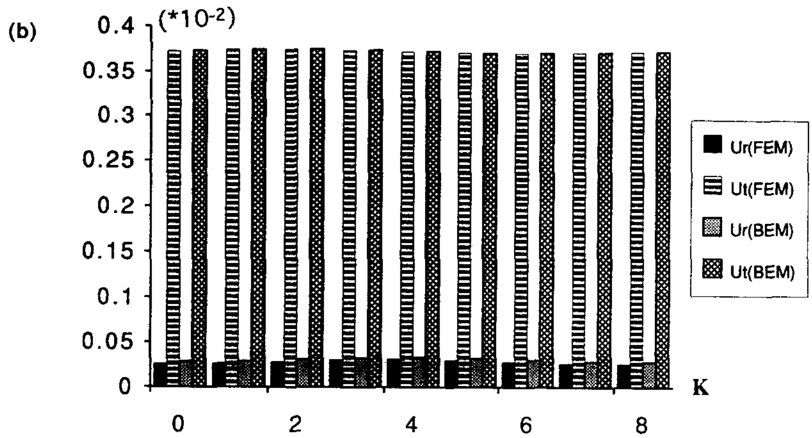


Fig. 5. (a) The basic symmetric part; (b) the BEM and FEM results of the radial displacements U_r and tangential displacements U_t on the inner boundary, $\theta = \pi k/32 - \pi/8$.

$N = 8$ was fully exploited; and case C, it was only regarded as a rotationally periodic system with the symmetry order $N = 4$, so that the corresponding basic region would contain two basic symmetric parts. These three cases gave the same results, but took very different CPU times and required different amounts of computer storage, as given in Table 1. Note that, in Table 1, $P1$ represents preprocessing, $P2$ solving equation, $P3$ postprocessing, the unit of time is the second, and storage means the number of the elements of real array needed. The special form of the load distributions for this problem resulted in only the first subproblem ($m = 0$) to be solved for each of cases B and C.

2. *An infinite plate with multi-holes*

As shown in Fig. 6, the problem deals with an infinite plate containing a centrally located hole, which is symmetrically surrounded by another four small holes and subjected to an uniaxial tension p at infinity. Because the symmetry order for this problem is $N = 4$, therefore it was divided into three subproblems. However the special form of the load distributions for this problem resulted in only two subproblems, i.e. first ($m = 0$) and third ($m = 2$) ones, to be solved. Figure 7 shows nondimensional stresses σ_{xx}/p and σ_{yy}/p along the X- and Y-axes.

CONCLUSIONS

Cyclic symmetry has been exploited for the analysis of rotationally periodic systems when using the BEM. Adopting a symmetry-adapted reference system, the block-circulant

Table 1. Case studies for comparison of CPU timing and storage with and without using symmetry

Case	$P1$	$P2$	$P3$	Storage
A	14.47	163.75	2.65	263,168
B	14.06	2.66	2.65	38,018
C	14.15	5.73	2.65	50,432

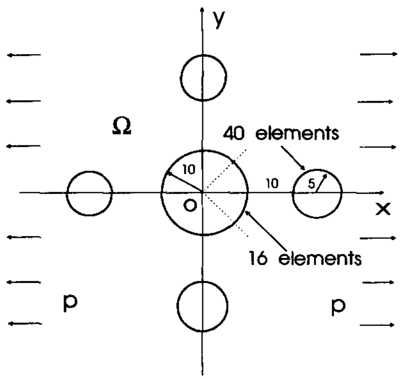


Fig. 6. An infinite plate with multi-holes subjected to uniaxial tension p at infinity.

matrix property of the corresponding coefficient matrices of the global BE equations has been proved, and then a partitioning algorithm was proposed.

A number of advantages can be gained by using cyclic symmetry in the BEM, which include:

(1) Instead of solving the original BE system equations, one only needs to solve a series of small subproblems. The sum of the dimensions of all these subproblems is equal to that of the original problem. Thus the computational efficiency is greatly increased. Since all the subproblems are independent, modern parallel processing computers can be profitably employed.

(2) The BE modeling can be limited only on any $1/N$ part of the whole computational boundary, and rather than to form the global coefficient matrices, only N small submatrices are needed for each coefficient matrix.

(3) The load distributions may be arbitrary, and the symmetry of the load conditions can be naturally exploited.

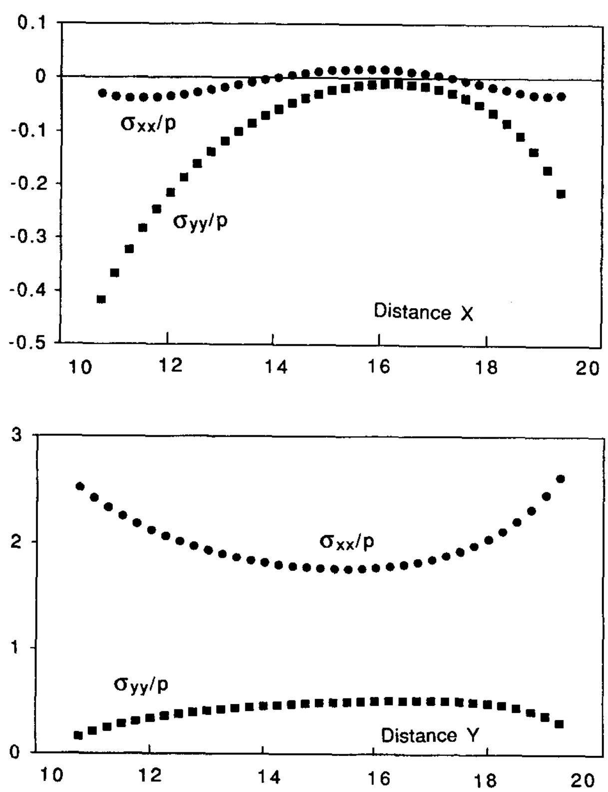


Fig. 7. Normalized stresses σ_{xx}/p and σ_{yy}/p along x and y axes.

The method can be readily extended and will be of great value to: three-dimensional problems, free vibration and dynamic problems and the engineering systems with other symmetric properties.

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